Chapter 2

Analysis of Circuits

2.1 INTRODUCTION AND DEFINITIONS

In the design and application of an instrument system a number of analog and digital circuits are used. Although it is not necessary to design complex electrical circuits in order to use instrument systems, it is important to understand the basic laws that govern the behavior of both ac and dc circuits. It is also important to analyze signals in order to describe the effect of the instrument system and its response on the quantity being measured. This chapter contains a review of the basic electronic concepts and laws that are useful in using and understanding modern instrumentation systems.

The analysis of circuits begins with defining the SI system of units, where the meter is the unit of length, the kilogram the unit of mass, and the second the unit of time. Other units of importance are absolute temperature in degrees Kelvin, relative temperature in degrees Celsius, and electric current in amperes. Quantities that will be used throughout this book, together with their standard symbol, units, and abbreviations, are defined in Table 2.1.

Brief definitions of each quantity follow:

**Force.** A force of 1 N causes a mass of 1 kg to accelerate at 1 m/s².

**Energy.** An object weighing 1 N receives 1 J of potential energy when it is elevated 1 m. Alternatively, a mass of 2 kg moving with a velocity of 1 m/s possesses 1 J of kinetic energy.

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<th>Table 2.1 Electrical Quantities Important in Instrumentation Systems</th>
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<td><strong>Quantity</strong></td>
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**Power.** Power represents the time rate at which energy is transformed. The transformation of 1 J of energy in 1 s represents an average power of 1 W. Instantaneous power is

\[ p = \frac{dw}{dt} \]  

(2.1)

**Charge.** An electric charge is the integral of the current with respect to time.

\[ q = \int i \, dt \]  

(2.2)

A charge of 1 C is transferred in 1 s by a current of 1 A.

**Current.** A current is the net rate of flow of positive charges.

\[ i = \frac{dq}{dt} \]  

(2.3)

A current of 1 A involves the transfer of a charge at the rate of 1 C/s.

**Voltage.** A charge of 1 C receives (or delivers) an energy of 1 J in moving through a voltage of 1 V. In general

\[ v = \frac{dw}{dq} \]  

(2.4)

**Electric Field Strength.** The electric field strength \( \vec{E} \) is defined by the magnitude and direction of the force \( \vec{F} \) on a unit positive charge in the electric field.

\[ \vec{F} = q\vec{E} \]  

(2.5)

It is easy to show that electric field strength is equal and opposite to the voltage gradient.

\[ \vec{E} = -\frac{dv}{d\ell} \]  

(2.6)

**Magnetic Flux Density.** A magnetic field develops in the region around a moving charge carrier or a current. The intensity of the magnetic effect is determined by

\[ \vec{B} = q\vec{v} \times \vec{E} \]  

(2.7)

where \( \vec{v} \) is the velocity of the charge \( q \), \( \vec{E} \) is the magnetic flux density, \( \times \) is the symbol representing the vector cross product.

A force of 1 N is developed by a charge of 1 C moving with a velocity of 1 m/s normal to a magnetic field with a flux density of 1 T.
Magnetic Flux. Magnetic flux is obtained by integrating the magnetic flux density over an area $A$.

$$\phi = \int \mathbf{B} \cdot d\mathbf{A}$$

(2.8)

where the symbol $\cdot$ represents the vector dot product.

The power $p$ and the energy $w$ transmitted in a circuit in terms of current $i$ and voltage $v$ are obtained as follows. From Eqs. 2.1, 2.3, and 2.4 we note that

$$p = \frac{dw}{dt} = \frac{dq}{dq} \frac{dq}{dt}$$

$$v = \frac{dw}{dq} \quad \text{and} \quad i = \frac{dq}{dt}$$

Therefore,

$$p = vi$$

(2.9)

$$w = \int p \, dt = \int vi \, dt$$

(2.10)

2.2 BASIC ELECTRICAL COMPONENTS

Analog and digital circuits are developed using several different components that affect the behavior of the current flow and the voltage at different locations in the circuit. This section introduces three of these basic components and gives the laws that govern the effect of the component on the circuit.

A. Resistance

The symbol for resistance is illustrated in Fig. 2.1a where the resistor $R$ is shown inserted in a circuit. Ohm’s law (in honor of Georg Ohm)

$$v = Ri$$

(2.11)

defines the relation between the voltage drop across the resistor and the current flow. When $v$ is expressed in volts and $i$ in amperes, $R$ is given in ohms $\Omega$.

The conductance $G$ of a component is the reciprocal of the resistance. Thus

$$G = \frac{1}{R}$$

(2.12)

where $G$ is expressed in terms of units known as siemens $S$.

B. Capacitance

The symbol for capacitance is illustrated in Fig. 2.1b where the capacitor $C$ is shown inserted in a simple circuit. Physically, the capacitor consists of two electrodes separated by a dielectric that serves as an insulator. When a voltage is applied, a positive voltage develops on the upper plate and a negative voltage develops on the lower plate. The capacitor stores the charge $q$ according to

$$q = Cv$$

(2.13)

and the energy stored $w$ is determined from Eqs. 2.13 and 2.4 as

$$w = \int_0^v C \, dv = \frac{1}{2} CV^2$$

(2.14)

When the voltage $v$ is constant, the charge on the capacitor is maintained and no current flows. However, if the voltage changes with time, current flows through the dielectric according to

$$i = C\frac{dv}{dt}$$

(2.15)

In Eqs. 2.13 through 2.15, the capacitance is expressed in farads F (named for Michael Faraday).

C. Inductance

The symbol for inductance is illustrated in Fig. 2.1c where the inductor $L$ is shown inserted in a simple circuit. Physically, the inductor is a multiturn coil of small-diameter wire. The coil has a very small resistance to steady current flow; however, when the current varies with time, a significant voltage drop develops across the coil as indicated by

$$v = L \frac{di}{dt}$$

(2.16)

where the inductance $L$ is expressed in henrys H (named for Joseph Henry).
The energy stored in the inductor is determined from Eqs. 2.10 and 2.16 as
\[ w = \int_0^t Li \, dt = \frac{1}{2} Li^2 \]  
(2.17)

2.3 KIRCHHOFF'S CIRCUIT LAWS

More than two centuries ago Gustav Kirchhoff developed two circuit laws that provide the foundations for network theory. The first is the current law, illustrated in Fig. 2.2a, which states that the algebraic sum of the currents flowing into a junction point at any instant is zero.
\[ \sum_i i = 0 \]  
(2.18)

The arrows representing the currents in Fig. 2.2 specify both magnitude and sign. Current flow into the junction is considered positive; flow away from the junction is considered negative.

The second of Kirchhoff’s circuit laws is the voltage law, which states that the algebraic sum of the voltages around a loop at any instant of time is zero.
\[ \sum_v v = 0 \]  
(2.19)

To show the application of Eq. 2.19, refer to the circuit loop presented in Fig. 2.2b and write the voltage change across each of the four legs of the loop as
\[ \sum_v v = v_{ba} + v_{cb} + v_{da} + v_{ad} = 0 \]
where \( v_{ba} \) indicates the voltage at point b measured with respect to point a. If \( v_{ba} \) is positive, point b is at a higher potential than point a. Now start at point a, go clockwise around the loop, and note
\[ \sum_v v = v_i - v_R - v_C + 0 = 0 \]
or
\[ v_i = v_R + v_C \]
at any instant of time. In this case, \( v_i \) represents the voltage source and \( v_R \) and \( v_C \) represent voltage drops in the direction of current flow.

2.4 DIODES, TRANSISTORS, AND GATES

The basic components of circuits, resistors, capacitors, and inductors, were reviewed in Section 2.2. Diodes, transistors, and gates are more advanced components, usually fabricated from semiconductors, that are employed with the basic components in most analog and digital circuits.

2.4.1 Diodes

A diode is a two-terminal component, shown symbolically in Fig. 2.3a. The ideal diode presents no resistance to current flow when a positive voltage (bias) is applied, as indicated in Fig. 2.3a. However, when the voltage is reversed (negative bias), the diode offers infinite resistance. Essentially, the diode acts as a selective switch, which is closed for reverse bias voltages and open for forward bias voltage. The voltage current characteristic of the diode, presented in Fig. 2.3b, is identical to that obtained with a selective switch.

In practice, most diodes are fabricated with a P/N junction in silicon and require a forward bias exceeding a threshold voltage of 0.6 V before conduction occurs.

2.4.2 Transistors

Transistors are semiconductor devices used either as amplifiers or as high-speed electronic switches. The most widely used amplifier is based on the bipolar junction transistor, illustrated in Fig. 2.4. The devices are planar; therefore, they can be fabricated by using lithographic methods in P- and N-doped silicon. They are extremely small, with areas of 10⁻⁴m².

The transistors are three-terminal devices, with the base represented by B, the collector by C, and the emitter by E. The theory of operation of the bipolar transistor is beyond the scope of this book. Basically, the transistor acts as a current amplifier because relatively small base currents \( i_B \) produce large collector currents \( i_C \). For example, when an NPN transistor is connected in a common emitter configuration with voltage sources and a resistive load \( R_L \), as shown in Fig. 2.5, the transistor acts to amplify the input signal \( i_I \). The signal current \( i_I \) causes a variation in the base current \( i_B \), which in turn produces a variation in the collector current \( i_C \), along the load line as shown in Fig. 2.5b. The time-varying part of the collector current represents the amplified output current that is drawn from the source \( V_C \) and the flow through the load resistance \( R_L \). The gain \( G \) is
\[ G = \frac{i_C}{i_I} \]  
(2.20)

![Figure 2.2](image-url)  
Figure 2.2 Circuit model for (a) Kirchhoff's current law, and (b) Kirchhoff's voltage law.

![Figure 2.3](image-url)  
Figure 2.3 (a) Symbol for a diode showing current flow with a positive bias. (b) Voltage-current characteristics of an ideal diode.
where \( i_C \) is the sinusoidal component of \( i_C \). The gain \( G \) depends on the base and collector characteristics of the transistor and \( V_{CC} \) and \( R_L \). Signal gains for a single transistor are in the range from 10 to 100.

Transistors are also employed as very high-speed electronic switches, which are open or closed depending on the voltage applied to the base. When operated as a switch, the transistor is connected into the simple circuit shown in Fig. 2.6a. Since all P/N junctions are reverse biased, practically no collector current flows and the transistor is operated in the cutoff region of Fig. 2.6b at point 1 when the input voltage (current) to the base is zero. At point 1 the collector current is small (5 \( \mu A \)), with an applied voltage \( V_{CC} = 5 \) V. This condition corresponds to a cutoff resistance of 1 MΩ, and the switch, whose contacts are the collector and emitter terminals, is open.

When a positive voltage is applied at the input, the base current increases (say, to 0.3 mA) and the operation of the transistor moves along the load line of Fig. 2.6b to point 2. At this point the transistor is operating in a saturated condition and the voltage drop \( V_{CE} \) across the transistor is very small. The collector current is about 30 mA at a saturation voltage of 0.3 V, which yields a switch resistance of about 10 \( \Omega \). In this state, the transistor is considered a closed switch.

### 2.4.3 Gates

In processing digital signals, the information is expressed as a digital code and is transmitted through logic operations, which transform and manipulate this information.

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**Figure 2.4** Representation of an NPN bipolar transistor. (a) Planar structure in silicon, and (b) circuit symbol.

**Figure 2.5** A basic NPN-transistor current amplifier. (a) Circuit diagram, and (b) operating characteristics.

**Figure 2.6** The transistor in a switching circuit, and (b) operating regions.
The logic gate is the device that controls the flow of information in a logic network. Although there are many different gates that perform specialized logic operations, all of these are made from elementary gates, namely AND, OR, and NOT gates.

The AND gate may be represented by the circuit shown in Fig. 2.7a, where two switches A and B are placed in the line from the source to the load. The voltage \( v_s \) is applied to the load only if switch A and switch B are both closed. The possibilities for the AND gate are listed in a truth table shown in Table 2.2. Note 0 is used to represent a false statement and 1 to represent a true statement. With regard to the voltage applied to the load, 1 indicates that it is true that \( v_s \) is applied to the load.

The OR gate is represented by the circuit shown in Fig. 2.7b, where two switches A and B are placed parallel to one another in the line between the voltage source and the load. When A is closed or when B is closed, the voltage is applied to the load \( (T = 1) \). The truth table for a two-switch OR gate is presented in Table 2.3.

The NOT gate, which is illustrated in Fig. 2.7c, is an inverter. In this case, the mechanical switch has been replaced by a transistor that is turned on (closed) by a positive input voltage. If the input signal to the transistor is, say, 0, the transistor acts as an open switch, no current flows, and the output voltage is \( v_o \) or 1. When the input signal goes to 1, the transistor conducts, acting like a closed switch, and the output is grounded, giving the low state, or 0. It is clear from this description that when the input is high (A), the output is low (\( \bar{A} \)), and changing the input to low (\( \bar{A} \)) results in an output that is high (A).

These basic gates are arranged in circuits to perform digital functions. A digital system is composed of many of these digital functions and may contain a million or more of the simple basic gates. The number of chips used to build the logic circuits depends on the scale of integration used to fabricate the circuits. With VLSI (very-large-scale integration) it is possible to place on the order of \( 10^6 \) gates on a single chip of silicon, thus permitting the development of extremely large digital systems with only 100 to 1000 chips.

### 2.5 DC CIRCUITS

In dc circuits, the current flow is constant with respect to time. This fact simplifies the circuit analysis because the voltage drop across an inductor is zero \( (di/dt = 0) \) and the current flow through a capacitor is zero \( (dv/dt = 0) \). The resistor is the only component that produces a voltage drop in accordance with Ohm’s law.

Consider resistors \( R_1 \) and \( R_2 \) arranged in series in a dc circuit, as shown in Fig. 2.8a. Kirchhoff’s voltage law, given by Eq. 2.19, and Ohm’s law, given by Eq. 2.11, yield

\[
v_s = v_{d1} + v_{d2} = iR_1 + iR_2 = iR_e
\]

where \( R_e = R_1 + R_2 \) is the equivalent closed-loop resistance as illustrated in Fig. 2.8a.

![Figure 2.7 Circuits and symbols for the three basic logic gates. (a) AND gate. (b) OR gate. (c) NOT gate.](image)

![Figure 2.8 Resistances in a circuit loop. (a) Series resistances, and (b) parallel resistances.](image)
Next, consider the parallel circuit illustrated in Fig. 2.8b and apply Kirchhoff’s current law, Eq. 2.18, to point A to obtain

\[ i_1 = i_1 + i_2 + i_3 \]  
\[ \text{(a)} \]

Substituting Eq. 2.11 into Eq. a gives

\[ \frac{v_a}{R_e} = \frac{v_a}{R_1} + \frac{v_a}{R_2} + \frac{v_a}{R_3} \]  
\[ \text{(b)} \]

Since \( v_a = v_d \), the equivalent resistance for a group of three parallel resistors is

\[ \frac{1}{R_e} = \frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3} \]  
\[ \text{(2.22)} \]

2.6 PERIODIC FUNCTIONS

When the current or voltage varies with time in a circuit, the signal has some type of waveform. The many different types of waveforms are all considered either periodic or transient. Periodic signals are repetitive and can be represented by sinusoidal functions or a series of sinusoidal components by means of Fourier analysis. Transient signals are one-time events; they do not repeat. They are more difficult to analyze and will be discussed in Chapter 10.

A special type of periodic function is the sinusoid (\( \sin\) not \( \cos\) because of frequency). Sinusoidal functions are important in describing the dynamic response of instrument systems where the ratio of the input voltage to the output voltage is a function of frequency. The sinusoid is also used extensively in Fourier analysis of other periodic signals that have a more complex waveform.

To illustrate two sinusoidal functions, consider a point rotating about a circle centered at point O in the \( xy \) plane as shown in Fig. 2.9. Since the line \( OR \) rotates with a constant magnitude \( A_0 \) in a counterclockwise direction with an angular velocity \( \omega \), the projection of \( OR \) onto the \( x \) axis gives the position of point \( P \) as

\[ x = A_0 \cos \omega t \]  
\[ \text{(2.23)} \]

and the projection of \( OR \) onto the \( y \) axis gives the position of point \( Q \) as

\[ y = A_0 \sin \omega t \]  
\[ \text{(2.24)} \]

The sinusoidal function is repetitive, as indicated in Fig. 2.9, with the values of \( x \) and \( y \) repeated every period \( T \) seconds. The angular frequency \( \omega \), the period \( T \) (s), and the cyclic frequency \( f \) (Hz) are related by the expression

\[ \omega = \frac{2\pi}{T} = 2\pi f \]  
\[ \text{(2.25)} \]

and from Eq. 2.25 it is evident that

\[ f = \frac{1}{T} \]  
\[ \text{(2.26)} \]

The velocities\(^1\) of points \( P \) and \( Q \), illustrated in Fig. 2.9, are given by

\[ v_P = \dot{x} = \frac{d}{dt}(A_0 \cos \omega t) = -A_0 \omega \sin \omega t = A_0 \omega \cos(\omega t + \frac{\pi}{2}) \]  
\[ \text{(2.27)} \]

\[ v_Q = \dot{y} = \frac{d}{dt}(A_0 \sin \omega t) = A_0 \omega \cos \omega t = A_0 \omega \sin(\omega t + \frac{\pi}{2}) \]  
\[ \text{(2.28)} \]

and the accelerations of points \( P \) and \( Q \) are

\[ a_P = \ddot{x} = -A_0 \omega^2 \cos \omega t \]  
\[ = A_0 \omega^2 \cos(\omega t + \pi) \]  
\[ \text{(2.29)} \]

\[ a_Q = \ddot{y} = -A_0 \omega^2 \sin \omega t \]  
\[ = A_0 \omega^2 \sin(\omega t + \pi) \]

Equations 2.27 and 2.28 show that the magnitudes of the velocities and accelerations of points \( P \) and \( Q \) can be obtained by multiplying the positions \( x \) and \( y \) by \( \omega \) and \( \omega^2 \), respectively. Note that there is a phase difference and the velocities and accelerations lead the displacements (positions) by \( \pi/2 \) and \( \pi \), respectively.

From Eqs. 2.23, 2.24, 2.27, and 2.28, the maximum values of the velocities and accelerations can be written as

\[ v_P = -\omega y \quad v_Q = \omega x \]  
\[ \text{(2.29)} \]

\[ a_P = -\omega^2 x \quad a_Q = -\omega^2 y \]

Since this motion is proportional to the displacement from a fixed point and the velocity \( v \) and acceleration \( a \) are directed toward that fixed point, the motion is classified as simple harmonic motion.

More complex waveforms that are periodic can be represented by a Fourier series of sinusoids. Thus,

\(^1\)The dot notation above a variable is used to indicate differentiation with respect to time. One dot indicates the first derivative, and two dots represent the second derivative.
\[ x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\omega t + \sum_{n=1}^{\infty} B_n \sin n\omega t \]  

(2.30)

where \( A_0, A_n, \) and \( B_n \) are the harmonic amplitudes and \( \omega \) is the fundamental frequency. If a sufficient number of terms are employed in the Fourier series representation, the periodic motions \( x, \dot{x}, \) or \( \ddot{x} \) can be accurately described with a sum of simple harmonic motions of frequencies that are multiples of the fundamental frequency (i.e., \( 2\omega, 3\omega, \cdots, n\omega \)).

A second method of analysis of signals uses phasors in the complex plane, where the phasor, projected on the real and imaginary axes, exhibits real and imaginary parts of the signal. Consider a phasor \( A \), shown in Fig. 2.10, and expressed in exponential form as

\[ A = A_0 e^{j\omega t} \]  

(2.31)

where \( j = \sqrt{-1} \). Recall the identity that

\[ e^{j\omega t} = \cos \omega t + j\sin \omega t \]  

(2.32)

Then Eq. 2.31 can be written as

\[ A = A_0 \cos \omega t + jA_0 \sin \omega t \]  

(2.33)

Note that the first term in Eq. 2.33 is real and the second term is imaginary. A graph of the complex plane, shown in Fig. 2.10, illustrates the phasor, the real and imaginary terms, and the angle \( (\omega t) \) that is taken as positive counterclockwise. A comparison of Figs. 2.9 and 2.10 indicates that the phasor represents the rotating line \( OR \), which provides an example of simple harmonic motion of a signal.

Differentiation of the phasor \( A \) to obtain \( dA/dt \) gives

\[ \dot{A} = j\omega A_0 e^{j\omega t} \]  

(2.34)

Note from Eq. 2.32 that \( j = e^{j\pi/2} \) and \( j^2 = -1 = e^{j\pi} \). Substituting the first of these equalities in Eq. 2.34 yields

\[ \dot{A} = \omega A_0 e^{j(\omega t + \pi/2)} \]

\[ = \omega A_0 \cos (\omega t + \pi/2) + j\omega A_0 \sin (\omega t + \pi/2) \]  

(2.35)

A comparison of Eq. 2.35 with Eq. 2.27 shows that the real part of \( \dot{A} \) corresponds to \( v_r \) and the imaginary part represents \( v_i \). Differentiating Eq. 2.34 to obtain \( d^2A/dt^2 \) gives

\[ \ddot{A} = -\omega^2 A_0 e^{j\omega t} \]  

(2.36)

By using Eq. 2.32 with Eq. 2.36, it is clear that

\[ \ddot{A} = \omega^2 A_0 e^{j(\omega t + \pi)} \]

\[ = \omega^2 A_0 \cos (\omega t + \pi) + j\omega^2 A_0 \sin (\omega t + \pi) \]  

(2.37)

A comparison of Eq. 2.37 with Eq. 2.28 indicates that the real part of \( \dot{A} \) corresponds to \( x_r \) and the imaginary part represents \( x_i \). The phase angle for \( A \) is \( \phi = \pi/2 \) and for \( \dot{A} \) is \( \phi = \pi \) relative to the reference phasor. These phase angles are leading as shown in the complex plane in Fig. 2.11. Note that leading phase angles are positive (counterclockwise) and lagging phase angles are negative (clockwise) with respect to the reference line.

The amplitude \( A_0 \) of the phasor is

\[ A_0 = \sqrt{(Re)^2 + (Im)^2} \]  

(2.38)

where

- \( Re \) is the amplitude of the real part
- \( Im \) is the amplitude of the imaginary part

The phase angle \( \phi \) is

\[ \phi = \tan^{-1} \frac{(Im)}{(Re)} \]  

(2.39)

The important advantages of using a phasor representation of sinusoidal motion include the ease of differentiating and integrating the exponential function and the presence of both magnitude and phase information. Differentiation is accomplished by multiplying by \( j\omega \), and integration is accomplished by dividing by \( j\omega \). Amplitude and phase angles for \( A, \dot{A}, \) and \( \ddot{A} \) can be determined easily by using Eqs 2.38 and 2.39. Because of these advantages, the exponential notation will be used throughout this book.

**Figure 2.10** Representation of the phasor \( A = A_0 e^{j\omega t} \) in a complex plane.

**Figure 2.11** Phase angles of \( \dot{A} \) and \( \ddot{A} \) relative to \( A \), showing velocity leading displacement by \( \pi/2 \) and acceleration leading displacement by \( \pi \).
2.7 AC CIRCUITS

The three fundamental electrical components used to describe the behavior of ac circuits are inductance $L$, resistance $R$, and capacitance $C$. These three components are illustrated individually in Fig. 2.12, where they are connected to a sinusoidal input voltage $v_s$. The voltage drop $v_d$ across each of these components was covered in Section 2.2. To determine the effect of each component in a circuit powered with an ac signal let

$$i = i_0 e^{jωt} \tag{2.40}$$

which can be substituted into Eqs. 2.11, 2.13, and 2.16 to give

$$v_d = jωL i_0 e^{jωt} = jωL i = Z_L i \quad \text{for the inductor}$$

$$v_d = R i_0 e^{jωt} = Ri = Z_R i \quad \text{for the resistor} \tag{2.41}$$

$$v_d = \left(\frac{i_0}{jωC}\right) e^{jωt} = \frac{i}{jωC} = Z_C i \quad \text{for the capacitor}$$

where $Z_L$, $Z_R$, and $Z_C$ are the impedances for the basic components. Thus,

$$Z_L = jωL \quad \text{AC Circuits} \quad 39$$

$$Z_R = R$$

$$Z_C = \frac{1}{jωC} = -\frac{j}{ωC}$$

From Eq. 2.42 it is evident that the inductance voltage leads the resistance voltage and current with a phase angle of $π/2$.

To illustrate the use of the impedance relations and to show elementary methods of ac circuit analysis, consider the circuit shown in Fig. 2.13. Applying Kirchhoff’s law, Eq. 2.19, and with Eqs. 2.11 and 2.13 gives

$$v_i = v_1(t) = iR + \frac{q}{C} \tag{a}$$

Since the output voltage $v_o = q/C$, Eq. a can be reduced to

$$v_1(t) = iR + v_o(t) \tag{b}$$

From Eq. 2.15 it is evident that

$$i = C \frac{dv_1(t)}{dt} = C \dot{v}_o(t) \tag{c}$$

Substituting Eq. c into Eq. b gives a first-order differential equation

$$RC \dot{v}_o(t) + v_o(t) = v_1(t) = v_i e^{jωt} \tag{2.43}$$

Let $v_o(t) = v_i e^{jωt}$ and substitute this relation into Eq. 2.43 to obtain

$$v_o(t) = \frac{1}{1 + jωRC} v_i e^{jωt} \tag{2.44}$$

Eliminating $j$ from the denominator of Eq. 2.44 gives

$$v_o(t) = \frac{1 - jωRC}{1 + (ωRC)^2} v_i e^{jωt} \tag{d}$$

\[\text{Figure 2.12 Elementary circuits illustrating voltage drop across the three basic components} \ L, \ R, \ \text{and} \ C.\]

\[\text{Figure 2.13} \ \text{An RC circuit with an output voltage} \ v_o(t) \ \text{representing the voltage across a capacitor.}\]
By using Eqs. 2.38 and 2.39 with Eq. 4, the output voltage across the capacitor in Fig. 2.13 can be represented by

\[ v_o(t) = \frac{v_i e^{j(\omega t - \phi)}}{\sqrt{1 + (\omega RC)^2}} = v_o e^{j(\omega t - \phi)} \]  

(2.45)

where

\[ v_o = \frac{v_i}{\sqrt{1 + (\omega RC)^2}} \]

and the phase angle \( \phi \) is

\[ \phi = \tan^{-1} \omega RC \]

Inspection of Eq. 2.45 shows that the amplitude \( v_o \) and the phase \( \phi \) of the output voltage are a function of a single combined term \( \omega RC \).

A second method of analysis for the circuit shown in Fig. 2.13 utilizes the impedances defined in Eq. 2.42. With this approach, the input and output voltages are written as sinusoids with the \( e^{j\omega t} \) notation. The voltage drop across a given element is taken as \( Z_i \). For example, the output voltage \( v_o(t) \) across the capacitor in Fig. 2.13 is given by Eq. 2.41 as

\[ v_o(t) = Z_C i \]  

(e)

But

\[ i = \frac{v_i e^{j\omega t}}{Z_R + Z_C} \]  

(f)

Substituting Eq. f into Eq. e yields

\[ v_o(t) = \frac{Z_C}{Z_R + Z_C} v_i e^{j\omega t} \]  

(g)

Next use Eq. 2.42 with Eq. g to obtain

\[ v_o(t) = \frac{1}{1 + j\omega RC} v_i e^{j\omega t} \]  

(2.46)

Comparison of Eq. 2.46 with Eq. 2.44 shows that the results are identical and that both methods can be used to determine the dynamic performance of ac circuits; however, the approach using impedances is easier and requires less time.

2.7.1 Impedance

In ac circuits, the impedance \( Z \) is related to a complex function that depends on the frequency of the signal. To show the impedance in the most general way, consider the circuit shown in Fig. 2.14. The circuit is driven with a sinusoidal voltage \( v_i(t) \).

If the voltage drops across the components are summed and equated to the supply voltage, then

\[ v_i = \left[ R + j \left( \frac{1}{\omega C} - \frac{1}{j\omega L} \right) \right] i_i \]  

(2.47)

Examination of Eq. 2.47 shows that a complex function with real and imaginary parts is involved in the relation between \( v_i \) and \( i_i \). If this complex function is divided into real and imaginary parts, as illustrated in Fig. 2.15, a more useful expression for \( v_i \) in terms of \( i_i \) is obtained.

\[ v_i = Z_i i_i \]  

(2.48)

where

\[ Z = R^2 + \left( \frac{1}{\omega L} - \frac{1}{\omega C} \right)^2 \]  

(2.49)

is the total impedance of the circuit.

Eqs. 2.48 and 2.49 define the amplitude of the voltage and the current, but the phase of the voltage relative to the current remains to be determined. Reference to Fig. 2.15 and Eqs. 2.39 and 2.47 shows the phase angle \( \phi \) as

\[ \phi = \tan^{-1} \left( \frac{\omega L - 1/\omega C}{R} \right) \]  

(2.50)

When the phase angle \( \phi > 0 \), as shown in Fig. 2.15, the voltage leads the current and the complete expression for \( v_i(t) \) is
\[ v_i(t) = \left( R^2 + \left( \frac{1}{\omega L} - \frac{1}{\omega C} \right) \right)^{1/2} i_j e^{j(\omega t + \phi)} \]  

(2.51)

Of particular importance is the effect of the frequency \( \omega \) on both the impedance \( Z \) and the phase angle \( \phi \).

### 2.8 Frequency Response Function

The frequency response function, often termed the FRF, for a circuit or instrument is defined as a ratio of output to input over a frequency range. Thus,

\[ H(\omega) = \frac{v_o(\omega)}{v_i(\omega)} \]  

(2.52)

where \( v_o(\omega) \) and \( v_i(\omega) \) are the frequency spectra of the output and the input signals. The frequency response function for the circuit of Fig. 2.13 can be written from Eq. 2.44 as

\[ H(\omega) = \frac{1}{1 + j\omega RC} = \frac{e^{-j\phi}}{\sqrt{1 + (\omega RC)^2}} \]  

(2.53)

where Eqs. 2.38 and 2.39 were used in the manipulation.

From Eq. 2.53 it is clear that the magnitude of the FRF is

\[ |H(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \]  

(2.54)

and the phase \( \phi \) is

\[ \phi = -\tan^{-1}\omega RC \]  

(2.55)

It is clear from Eq. 2.54 that the frequency response function \( H(\omega) \) gives the ratio for the amplitude of the output voltage to the input voltage, and Eq. 2.55 gives the phase shift \( \phi \) of the output voltage relative to the input voltage. The negative phase angle indicates that the output signal lags behind the input signal.

The frequency response function and other parameters are often expressed as a relative number in terms of decibels \( N_{dB} \). The number of decibels is defined as

\[ N_{dB} = 10\log\left( \frac{p}{p_r} \right) \]  

(2.56)

where

- \( p \) is the measured power
- \( p_r \) is a reference power

The decibel can also be expressed in terms of a voltage ratio by substituting \( p = \frac{v^2}{R} \) into Eq. 2.56 to obtain

\[ N_{dB} = 20\log\left( \frac{v}{v_r} \right) \]  

(2.57)

When describing the dynamic behavior of a measurement system in \( N_{dB} \) it is essential to specify the reference quantity \( p_r \) or \( v_r \).

The magnitude and phase of \( H(\omega) \) are represented graphically on a Bode diagram, where \( |H(\omega)| \) and \( \phi \) are shown individually as functions of \( \omega RC \). The magnitude \( |H(\omega)| \) is represented in terms of decibels on a Bode diagram and the \( \omega RC \) parameter is shown on a log scale. To illustrate the construction of a Bode diagram, rewrite Eq. 2.54 in terms of decibels by using Eq. 2.57 to obtain

\[ N_{dB} = 20\log\left( \frac{1}{\sqrt{1 + (\omega RC)^2}} \right) \]

\[ = -10\log\left( \frac{1}{1 + (\omega RC)^2} \right) \]  

(2.58)

The phase \( \phi \) is given directly by Eq. 2.55.

The Bode diagrams correspond to Eqs. 2.58 and 2.55 are shown in Fig. 2.16. The magnitude of \( H(\omega) \) is down 3 dB when \( \omega RC = 1 \), and when \( \omega RC \gg 1 \) the magnitude decays linearly at 20 dB per decade. This example illustrates that the Bode diagrams provide a visual representation of a wide dynamic range of circuit or instrument characteristics. They are thus useful in determining the behavior of instruments in application to dynamic measurements.

![Figure 2.16 Bode diagram (a) magnitude, and (b) phase for the RC circuit shown in Figure 2.13.](image-url)
2.9 SUMMARY

In the design and application of an instrumentation system, a number of analog and digital circuits are used. An understanding of the basic laws that govern the behavior of these circuits is required to make effective use of the systems. In this chapter, brief definitions are provided for all of the electrical quantities encountered in the remaining chapters of the book together with a listing of their standard symbols, units, and abbreviations. Fundamental relationships among these electrical quantities are summarized in Table 2.4.

Three basic components (resistors, capacitors, and inductors) affect the behavior of current flow and voltage in all electrical circuits. Ohm’s law defines the relation between voltage drop across a resistor and current flow. A capacitor stores electric charge. When a constant voltage is applied to a capacitor, the charge is maintained and no current flows. When the voltage changes with time, current flows. An inductor exhibits only a very small resistance to a steady flow of current. When the current varies with time, however, a significant voltage drop develops across an inductor.

Diodes, transistors, and gates are advanced components that are employed with the basic components in most analog and digital circuits. A diode acts as a selective switch. It offers no resistance to current flow if a positive voltage is applied, and infinite resistance to current flow if the voltage is reversed. Transistors are used as amplifiers or as high-speed switches. Signal gains for a single transistor can range from 10 to 100. Gates are used in logic networks to control the flow of information. Circuits designed to perform specialized operations may contain large numbers of AND, OR, and NOT gates on a single silicon chip.

Kirchhoff’s two laws, the voltage law and the current law, provide the foundations for circuit analysis. Analysis procedures are presented for both dc and ac circuits.

<table>
<thead>
<tr>
<th>Table 2.4 Summary of Basic Relations</th>
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</thead>
<tbody>
<tr>
<td><strong>Element</strong></td>
</tr>
<tr>
<td>Resistance (Conductance)</td>
</tr>
<tr>
<td>Inductance</td>
</tr>
<tr>
<td>Capacitance</td>
</tr>
<tr>
<td>Short circuit</td>
</tr>
<tr>
<td>Open circuit</td>
</tr>
<tr>
<td>Voltage source</td>
</tr>
<tr>
<td>Current source</td>
</tr>
</tbody>
</table>

In dc circuits, where the current flow is constant, the resistor is the only component producing a voltage drop. In ac circuits, the current varies with time; therefore, voltage drops develop across resistors, capacitors, and inductors. Impedance in an ac circuit was defined and the magnitude and phase of the voltage drops with respect to the current was determined.

Signal analysis is an important part of any experimental investigation. Periodic signals are repetitive; transient signals are one-shot events that do not repeat. Sinusoidal functions, which are a special type of periodic function, are very important in describing the dynamic response of instrument systems where the ratio of the input voltage to the output voltage is a function of frequency. Also, the sinusoid is used extensively in the Fourier analysis of other periodic signals with a more complex waveform.

A second method of analysis of signals uses phasors in the complex plane. The advantages of using a phasor representation of sinusoidal signals include ease of differentiating and integrating and the presence of both magnitude and phase information. Differentiation is accomplished simply by multiplying by \( jo \); integration is accomplished by dividing by \( jo \).

The frequency response function for a circuit or for an instrument gives the amplitude ratio of output voltage to input voltage and the phase shift \( \phi \) of the output voltage relative to the input voltage. The frequency response function and other parameters are often expressed as a relative number in terms of decibels \( N_{dB} \). Bode diagrams provide a visual representation of a wide dynamic range of circuit or instrument characteristics. Thus, they are useful in determining the behavior of instruments in application to dynamic measurements.

REFERENCES


EXERCISES

2.1 List the symbol, units, and abbreviations for
(a) Force, charge, and electric field strength
(b) Energy, current, and magnetic flux density
(c) Power, voltage, and magnetic flux

2.2 For a resistance \( R = R_o \), placed across a voltage supply \( V_s \), determine the power dissipated as \( V_s \) is increased from 0 to 1000 V. Prepare a graph showing these results if
(a) \( R_o = 50,000 \, \Omega \)
(b) \( R_o = 100,000 \, \Omega \)
(c) \( R_o = 250,000 \, \Omega \)
(d) \( R_o = 500,000 \, \Omega \)
2.3 A voltage \( v_o \) is switched across a capacitor \( C \) at time \( t = 0 \). Derive an equation that describes the current flow with time during charging of the capacitor if the voltage source has an internal resistance \( R \).

2.4 Using the results of Exercise 2.3, prepare a graph showing \( i(t) \) if

(a) \( v_o = 10 \) V and \( C = 10 \) \( \mu F \)
(b) \( v_o = 5 \) V and \( C = 200 \) pF
(c) \( v_o = 100 \) V and \( C = 1 \) F
(d) \( v_o = 3 \) V and \( C = 1500 \) pF

and the source resistance is 10 \( \Omega \).

2.5 Determine the charge and energy stored by the capacitor in each case listed in Exercise 2.4.

2.6 An ac voltage described by the expression \( v = a \sin 2\pi f t \) is switched across an inductor \( L \) at \( t = 0 \). Derive an expression for the current flow with time through the inductor.

2.7 Using the results of Exercise 2.6,

(a) prepare a graph showing \( v(t) \) on the abscissa and \( i(t) \) on the ordinate. Let

(1) \( f = 60 \) Hz, \( L = 10 \) mH, \( a = 10 \) V
(2) \( f = 1 \) MHz, \( L = 10 \) \( \mu F \), \( a = 5 \) V

(b) What is the shape of the curve you have plotted?
(c) Is the shape stationary with respect to time?

2.8 Determine the energy stored in the inductor for the two cases given in Exercise 2.7.

2.9 In your own words describe Kirchhoff’s first law. Indicate why it is an important principle in circuit analysis.

2.10 Repeat Exercise 2.9 for Kirchhoff’s second law.

2.11 Sketch the symbol for a diode. Explain what is implied by positive bias voltage and negative bias voltage.

2.2 A voltage \( v_1 = 10 \sin(120\pi t) \) is applied to a diode as shown in Fig. E2.12. Prepare a graph showing \( v_2(t) \).

2.3 What are the two primary applications for transistors?

2.4 Using the characteristic curves for a transducer given in Fig. 2.5b, determine the maximum and minimum values of the collector current \( i_c \) if \( i_1 = 0.2 - 0.1 \sin \omega t(10)^{-3} \). Note: \( V_{cc} = 10 \) V and \( R_L = 400 \) \( \Omega \).

2.15 Sketch the circuit for an electronic switch that employs a bipolar transistor. Use your own words to describe its operation.

2.16 For a transistor employed as a switch, as illustrated in Fig. 2.6, determine the current flow through the switch when it is open and when it is closed. Compare these values to those obtained using a mechanical switch. Are these differences important? If so, when?

2.17 Sketch the symbol and write a truth table for the AND gate and the OR gate.

2.18 Sketch the circuit using transistors as switches for the following basic gates:

(a) AND
(b) OR
(c) NOT

2.19 With reference to Fig. 2.8, verify the relations for \( R_e \) given in Eqs. 2.21 and 2.22.

2.20 For a displacement given by \( x = A_0 \cos(\omega t + \phi) \), show that the velocity \( \dot{x} \) and the acceleration \( \ddot{x} \) exhibit a phase difference with respect to the displacement of \( \pi/2 \) and \( \pi \), respectively.

2.21 For the sawtooth function shown in Fig. E2.21, write \( y(t) \) in terms of a Fourier series expansion.

2.22 A quantity \( Q = A_0 e^{j(\omega t + \phi)} \). (a) Find \( \dot{Q} \) and \( \ddot{Q} \). (b) What is the effect of the phase angle \( \phi \) on these unknowns?

2.23 A phasor has a real part \( Re = R \) and an imaginary part \( Im = \omega L - 1/\omega C \). Find the amplitude of the phasor and its phase angle.

2.24 For the circuit shown in Fig. 2.13, (a) verify Eq. 2.45. (b) Prepare a graph showing the amplitude of \( v_o/v_i \) as a function of \( \omega RC \). (c) What happens to the impedance \( Z_e \) as \( \omega \) becomes very large?

2.25 For the circuit shown in Fig. E2.25, derive an expression for \( v_o(t) \). Define the amplitude and the phase angle in this expression.

2.26 For the circuit shown in Fig. E2.26 determine the impedance and the phase angle between the current and the voltage.

2.27 For the circuit shown in Fig. 2.14, prepare a graph of \( Z(\omega) \) if \( L, C \), and \( R \) have the following values

\[
L(\mu H) \quad C(\mu F) \quad R(\Omega)
\]

(a) 0.01 0.50 10,000
(b) 0.05 0.20 1000
(c) 0.10 0.10 500
(d) 0.20 0.20 200
(e) 0.50 0.50 100

2.28 Determine the magnitude \( H(\omega) \) and the phase angle \( \phi \) of the frequency response function for the circuit shown in Fig. 2.13 with the values of \( R \) and \( C \) listed in Exercise 2.27.

2.29 Determine the decibel equivalents for the following ratios of \( p/p_r \) and \( v/v_s \):

\[
p/p_r \quad v/v_s
\]

(a) 1000 15
(b) 2.0 -0.001
(c) 0.003 3000

2.30 Sketch the circuit for Exercise 2.15.