Finite Element Analysis of Stress
Induced Damage in Ceramics

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To

my wife Dorita and

my children Leandro and Sebastián
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Ceramic materials have exceptional properties such as low density, high stiffness, strength, and melting temperature. Nevertheless, their applicability is very restricted due to their low fracture toughness.

Damage in polycrystalline ceramics during fabrication is due mainly to microcracking around grains boundaries. Their subsequent strength and fracture behavior are controlled by these microcracks. Sintering and hot pressing at high temperatures, followed by cooling, causes residual stresses due to the thermal anisotropy of the grains (Ortiz and Molinari [17]). These stresses can generate microcracking along grains boundaries, depending upon the grain sizes (Rice and Pohanka [21]). The residual stresses play an important role in subsequent loading as reported by Fu and Evans [9] and as will be shown in this work. The direct effect of the residual stresses considered here is the introduction of permanent strains upon stress relaxation at the crack faces. Other mechanisms proposed for explaining the permanent strains generated during loading include frictional sliding of crack surfaces and crack closure. The latter mechanism is relevant in ceramics that can undergo transformation under stress, for instance MgO-PSZ (Magnesio-partially stabilized Zirconia) which contains metastable particles that transform from tetragonal to monoclinic unit cells. The transformation strain also plays an important role in cyclic compression fracture, as pointed out by Suresh and Brockenbrough [25].

A complete study of the loss of stiffness in a solid containing a dilute population of microcracks and subjected to quasi-static tension was carried out by Budiansky and O'Connell [3]. In the present work, the case of dynamic loading is addressed
in the same spirit. Moreover, an assessment of the influence of the parameters characterizing microcracking is performed. The purpose of the study is to develop an understanding of the microcracking phenomenon as well as the role of the loading rate in the structural response of the material.

In computational studies of problems that involve strain-softening, e.g. due to microcracking as in this work, oscillations in the solution beyond those corresponding to Gibb’s phenomenon, are often observed. These oscillations have been minimized in this work by introducing a remeshing scheme that maintains the ideal condition for information to propagate across one element at each time step.
CHAPTER I

Model Definition

I. Background

A plate impact experiment to study the evolution of damage in ceramics was introduced by Raiser et al. [20]. In this thesis the numerical simulation of this experiment is investigated.

We begin with a summary of the linear wave propagation problem given in [20], which describes the loading history to be used in the analysis. The time-distance (t-x) diagram for the soft recovery experiment is shown in Fig. 1.

The experiment uses a star flyer plate that impacts a ceramic specimen and a momentum trap plate. The impact causes two compressive waves to emanate from the impacted surface (x=0). One wave travels into the specimen causing a compressive stress. The other travels into the star flyer, reflects from the free surface and becomes a tensile pulse which unloads the compressive stress. This unloading wave proceeds into the specimen and removes the compressive stress there. The specimen, therefore, is subjected to a compressive pulse of a duration equal to the wave’s round-trip travel time through the star flyer. When the first part of this pulse reaches the rear surface of the specimen, the gap between the specimen and the momentum trap causes the surface of the specimen to act like a free surface for a period of time equal to the time it takes the particle velocity to close the gap. The effect of the reflection from this surface is to produce a tensile pulse which crosses the compression region (2) and causes a zero stress
Figure 1. Time-Distance diagram for soft recovery experiment [20].
state (3). This tensile pulse then propagates through the specimen causing tension over the region (4). The pulse then reflects from the impact surface of the specimen, becomes a compressive pulse, and proceeds through the specimen once more (6) before going into the momentum trap. The initial compressive pulse, minus the reflected part which caused the tensile pulse, travels through the momentum trap, reflects from the free surface, and becomes a tensile pulse. When this tensile pulse reaches the interface between the specimen and the momentum trap, the latter separates from the specimen. The momentum trap flies off, leaving the specimen unstressed and with zero momentum.

II. Governing Equations

The equations of motion and compatibility are:

\[ \sigma_x = \rho v_t \quad \text{in} \quad (0, L) \]
\[ v_x = \varepsilon_t \]

where \( \sigma \) is the stress, \( v \) is the particle velocity, \( \varepsilon \) is the strain, \( \rho \) is the mass density, and \( L \) is the length of the ceramic plate.

The boundary conditions are:

at \( x = 0 \):

\[ \sigma(0, t) - \rho c_1 \ v(0, t) = -\rho c_1 V_0 \quad \text{if} \quad t \leq t_1 \]

at \( x = L \):

\[ \sigma(L, t) = \begin{cases} 
0 & \text{if} \quad t \leq t_2 ; \\
-\rho c_2 \ v(L, t) & \text{if} \quad t_2 < t < t_3 ,
\end{cases} \]
where $V_0$ is the impact velocity, $t_1$ is the time of contact between the flyer and the ceramic plate, $t_2$ is the time at which contact between the ceramic plate and the momentum trap takes place, $t_3$ is the time at which the reflection from the rear surface of the momentum trap arrives at the interface between the specimen and the momentum trap, $\rho c_1$ and $\rho c_2$ are the impedances of the flyer and the momentum trap plate, respectively.

The above equations are obtained using the characteristic equations

$$\sigma \pm (\rho c)v = \text{constant} \quad \text{along} \quad \frac{dx}{dt} = \mp c$$

(4)

where $c$ is the longitudinal wave speed of the material. The characteristic with positive slope is used for the derivation of the boundary condition at $x = 0$, while the characteristic with negative slope is used at $x = L$. The idea is to use information from earlier times.

The amplitude of the stresses at the regions (2), (4) and (6), see Figure 1, is governed by the impact velocity $V_0$, while the duration of the tensile pulse is controlled by the size of the gap.

III. Strain-Stress Relations

To derive the constitutive equation, let us consider penny-shape microcracks perpendicular to the direction of loading. Then the strain-stress relations are:
\[
\begin{align*}
\varepsilon_1 &= \frac{\sigma_1}{E_s} - \frac{\nu(\sigma_2 + \sigma_3)}{E} + \varepsilon_1^T \\
\varepsilon_2 &= \frac{\sigma_2}{E} - \frac{\nu(\sigma_1 + \sigma_3)}{E} \\
\varepsilon_3 &= \frac{\sigma_3}{E} - \frac{\nu(\sigma_1 + \sigma_2)}{E},
\end{align*}
\tag{5}
\]

where \(E_s\) is a secant modulus which accounts for the decrease in stiffness due to the damage in the ceramic, \(\varepsilon_1^T\) is the transformation or permanent strain, \(E\) and \(\nu\) are the Young’s and Poisson’s moduli of the material which comprises the matrix.

For the case of uniaxial straining in the 1-direction, \(\varepsilon_2 = \varepsilon_3 = 0\), one finds that
\[
\varepsilon_1 = \left( \frac{1}{E_s} - \frac{2\nu^2}{1 - \nu} \frac{1}{E} \right) \sigma_1 + \varepsilon_1^T = \frac{\sigma_1}{E_{eff}} + \varepsilon_1^T. \tag{6}
\]

In order to obtain an expression for \(E_s\), the following energy equilibrium argument is invoked. Let the body be in a state of uniaxial tension, \(\sigma_1 = \sigma, \sigma_2 = \sigma_3 = 0\). Assume that
\[
\frac{\sigma^2}{2E_s} = \frac{\sigma^2}{2E} + \mathcal{E}N, \tag{7}
\]
where \(N\) is the number of penny-shape cracks per unit volume, and \(\mathcal{E}\) is the strain energy of an isolated penny shaped crack in an infinite medium. The left hand side of eq. (7) represents the strain energy per unit volume of the body with its effective properties, while the right hand side is the superposition of the strain energy per unit volume, in the absence of any degradation of the material, and the energy of the nonuniform strain fields associated with the penny shaped cracks.

The expression for \(\mathcal{E}\) is (Sneddon [24])
\[
\mathcal{E} = \frac{8}{3}(1 - \nu^2) a^3 \sigma^2 \frac{E}{E}, \tag{8}
\]

\[ -7 - \]
where \( a \) is the crack radius.

After substituting (8) into (7) one obtains

\[
\frac{1}{E_s} = \frac{1}{E} + \frac{16}{3} \frac{(1 - \nu^2)Na^3}{E}.
\]  

(9)

It should be emphasized that this analysis assumes a dilute concentration of microcracks, and thus neglects the interaction between them.

IV. Dynamic Crack Growth Criterion

The equation of evolution for the crack radius, \( a = a(x, t) \), at a fixed position can be derived from the following dynamic criterion for crack growth

\[
K_{ID}(a, \dot{a}) = K_{IC},
\]  

(10)

where \( K_{IC} \) is the fracture toughness of the ceramic, and \( K_{ID} \) is the dynamic stress intensity factor. The dynamic stress intensity factor is a function of the crack radius, \( a \), and the crack velocity, \( \dot{a} \). For a semi-infinite crack, it is given by Freund [8] as

\[
K_{ID}(a, \dot{a}) = k(\dot{a}) K_{IS}(a),
\]  

(11)

where \( K_{IS} \) is the static stress intensity factor, and \( k(\dot{a}) \) is a function of the crack velocity and the Rayleigh wave speed \( c_R \). Equation (11) is adopted in this work as a model crack tip equation of motion, with
\[ k(\dot{a}) = \left(1 - \frac{\dot{a}}{c_R}\right)^{-\frac{1}{m}}. \] (12)

The plot of this function for different values of \( m \) is shown in Fig. 2.a.

![Graph showing the ratio between dynamic and static stress intensity factors as a function of crack velocity.](image)

**Figure 2.a.** Ratio between dynamic and static stress intensity factors as a function of crack velocity.

The static stress intensity factor for remote loading \( \sigma \) perpendicular to a penny-shape crack of radius \( a \) in an infinite body is given by Sneddon [24] as

\[ K_{IS}(a) = 2\sigma \sqrt{\frac{a}{\pi}}. \] (13)

It is known that, when a stationary crack of length \( 2a \) is subjected to a uniform tensile pulse at normal incidence, the stress intensity factor increases up to a level of approximately 1.2 \( K_{IS} \) until diffraction waves from the opposite end of the crack
reach the crack tip under consideration. For subsequent times, $K_{ID}$ tends asymptotically to the value $K_{IS}$ (see Fig. 2.b). This result demonstrates the existence of an incubation time prior to the attainment of the critical condition (10). In the present study, this effect is not considered since the incubation time is of the order of $2a/c$ and is smaller than the rise time of the applied pulse.

![Graph](image)

**Figure 2.b.** Ratio between dynamic and static stress intensity factors as a function of normalized time.

Combining equations (10) through (13) one derives the final expression for the crack tip velocity

$$
\dot{a}(t) = c_R \left( 1 - \left( \frac{K_{IC}}{2\sigma} \right)^m \left( \frac{\pi}{a} \right)^{\frac{m}{2}} \right).
$$

(14)

In the above formula, $(.) = 0$ if the argument is negative.
V. Transformation Strain

As mentioned in the introduction, the residual stresses incurred during fabrication play an important role in the subsequent behavior of the material. In this section, an expression for the transformation strain, resulting from stress relaxation at the crack faces, is derived. According to Eshelby [6] the sequence of states shown in Figure 3 characterize the desired transformation strain for a homogeneous body containing a self-equilibrated state of stress $\sigma_R$, i.e. $\int_{\partial V} \sigma_{ij}^R n_j dS = 0 \quad \forall \partial V$.

By considering the energy release in state (b), for the case of a flat cavity of radius $a$ in an infinite body, one has

$$\mathcal{E} = \frac{8}{3} \frac{(1 - \nu^2)}{E} a^3 \sigma_R^2. \quad (15)$$

One can obtain the change in volume by direct differentiation of the energy release $\mathcal{E}$ with respect to the residual stress $\sigma_R$, which yields

$$\Delta V^T = \frac{16}{3} \frac{(1 - \nu^2)}{E} a^3 \sigma_R. \quad (16)$$

This result holds for each microcrack independently of its position in the solid. Therefore, for varying microcrack sizes and arbitrary distribution of residual stresses, one can consider the average value $\langle a^3 \sigma_R \rangle$ and obtain the desired homogenized transformation strain after multiplication of $\varepsilon_1^T$ by the number of microcracks per unit volume, viz.

$$\varepsilon^T(a) = \frac{16}{3} \frac{(1 - \nu^2)}{E} N \langle a^3 \sigma_R \rangle. \quad (17)$$
Figure 3. Schematic representation of the states characterizing the transformation strain.
The value of the residual stresses due to the thermal anisotropy of the grains in ceramic materials is described and estimated analytically by Ortiz and Molinari [17].
CHAPTER II

Numerical Implementation

I. Numerical Solution of the Wave Equation

A standard finite-element discretization for the spatial variable and an explicit-implicit time integration algorithm is used (see Hughes [12], Oden and Carey [16], Ortiz et al. [18-19]). The steps in the algorithm are the following:

- Predictor phase:

\[
\hat{d}_{n+1} = d_n + \Delta t v_n + (1/2 - \beta) \Delta t^2 a_n
\]
\[
\hat{v}_{n+1} = v_n + (1 - \gamma) \Delta t a_n,
\]

where \( d \) is the displacement vector, \( v \) is the velocity vector, \( a \) is the acceleration vector, \( \Delta t \) is the time step increment, \( \gamma \) and \( \beta \) are the standard Newmark parameters, and the indices \( n \) and \( n + 1 \) refer to two subsequent times.

- Equation solving phase:

\[
\hat{a}_{n+1}^e = [M^e + \gamma \Delta t C^e + \beta \Delta t^2 K^e]^{-1} \left[ f_{n+1}^e - K^e \hat{a}_{n+1}^e - C^e \hat{v}_{n+1}^e \right]
\]
\[
a_{n+1} = M^{-1} \sum_{e} M^e \hat{a}_{n+1}^e \quad \text{(averaging rule)}
\]

with \( e = 1, \ldots, \# \) elements,

where \( M^e \) is the mass matrix of the element, \( K^e \) its stiffness matrix, \( C^e \) its damping matrix, \( f^e \) is the vector of nodal forces of the element and \( M \) is the global mass matrix.
- Corrector phase:
\[
\begin{align*}
\mathbf{d}_{n+1} &= \mathbf{d}_{n+1} + \beta \Delta t^2 \mathbf{a}_{n+1} \\
\mathbf{v}_{n+1} &= \mathbf{v}_{n+1} + \gamma \Delta t \mathbf{a}_{n+1}.
\end{align*}
\] (20)

The predictor and corrector steps in the scheme are identical to those of Newmark's method. However, the equation solving phase is designed to allow concurrency in the computations while achieving unconditional stability. During the predictor phase, accelerations are computed at the element level and then compatibility between elements is restored using a mass average rule. The algorithm introduces some numerical damping when \(\gamma \neq 1/2\) (see [18] and [12]).

The damping matrix \(\mathbf{C}^e\) introduced in equation (19) takes into consideration the boundary condition at \(x = L\) of the impact tests (see equation (3)). The expression for \(\mathbf{C}^e\) for the element located at the end of the specimen and in contact with the momentum trap plate has a non-zero entry given by the impedance of this plate,

\[
\mathbf{C}^e = \begin{pmatrix} 0 & 0 \\ 0 & \rho c_2 \end{pmatrix}.
\] (21)

This boundary condition, simulating the contact between the specimen and the momentum trap plate, has the advantage of simplicity. As a drawback, large oscillations in stresses tend to develop. Since the equation of evolution (14) has a strong dependence on stress, the \textit{spurious oscillations} may severely corrupt the solution. Here, the troublesome oscillations are eliminated by means of a spatial three point filter.
II. Optimal Choice of Parameters

The correct choice of parameters to be used in the algorithm such that no artificial reverberations appear is revealed by the following analysis.

Consider the harmonic wave given by,

\[ d_j^n = A e^{i(\omega n \Delta t + kj \Delta x)} \]
\[ v_j^n = B e^{i(\omega n \Delta t + kj \Delta x)} \]
\[ a_j^n = C e^{i(\omega n \Delta t + kj \Delta x)}, \]

where A, B and C are arbitrary constants, \( \omega \) is the wave-frequency, \( k \) is the wave-number, \( \Delta x \) is the element size, and \( \Delta t \) is the time step increment. The indices \( j \) and \( n \) refer to position and time, respectively.

At the highest frequency allowed by the mesh, i.e. \( \omega = \pi / \Delta t \) and \( k = \pi / \Delta x \), one application of the predictor equations gives,

\[ \hat{d}_j^{n+1} = [A + B \Delta t + (\frac{1}{2} - \beta)\Delta t^2 C]e^{i\pi(j+n)} \]
\[ \hat{a}_j^{n+1} = [B + (1 - \gamma) \Delta t C]e^{i\pi(j+n)}. \]

From equations (19.a), the acceleration at node \( j \) is given by

\[ a_j^{n+1} = \frac{c^2 / \Delta x^2}{1 + \beta(2r)^2} (\hat{d}_{j-1}^{n+1} + \hat{d}_{j+1}^{n+1} - 2\hat{d}_j^{n+1}). \]

Solving for \( a_j^{n+1}, v_j^{n+1}, d_j^{n+1} \) and imposing the condition that the result be exact one gets,
\[ a_j^{n+1} = \frac{c^2/\Delta x^2}{1 + \beta(2r)^2} \left[ A + B\Delta t + \left( \frac{1}{2} - \beta \right) \Delta t^2 C \right] e^{i\pi(j+n)}(e^{-i\pi} + e^{i\pi} - 2) \]
\[ \equiv Ce^{i\pi(j+n)}e^{i\pi} \]

\[ v_j^{n+1} = [B + (1 - \gamma)\Delta t C] e^{i\pi(j+n)} + \frac{\gamma rc/\Delta x}{1 + \beta(2r)^2} \left[ A + B\Delta t + \left( \frac{1}{2} - \beta \right) \Delta t^2 C \right] e^{i\pi(j+n)}(e^{-i\pi} + e^{i\pi} - 2) \equiv Be^{i\pi(j+n)}e^{i\pi} \]

\[ d_j^{n+1} = \left[ A + B\Delta t + \left( \frac{1}{2} - \beta \right) \Delta t^2 C \right] e^{i\pi(j+n)} + \frac{\beta r^2}{1 + \beta(2r)^2} \left[ A + B\Delta t + \left( \frac{1}{2} - \beta \right) \Delta t^2 C \right] e^{i\pi(j+n)}(e^{-i\pi} + e^{i\pi} - 2) \equiv Ae^{i\pi(j+n)}e^{i\pi}. \]

(25)

Solving these three equations one obtains the desired relation between the parameters \( \beta, \gamma \) and the Courant number \( r = c\Delta t/\Delta x \),

\[ \beta = \frac{2\gamma r^2 - 1}{4r^2}, \]

(26)

where \( c \) is the velocity of wave propagation of the medium, which in turn is function of the secant modulus and the mass density.

III. Remeshing Procedure

The additional feature of remeshing is introduced in the model in order to have at all times the ideal condition of \( r = 1 \) in all the elements of the mesh. Physically it means that the wave advances one element per time step. This suggests the use
of the velocity of propagation of the wave as the main variable in the procedure.

The main steps in this scheme are the following (see Figure 4):

i) Computation of the integral

\[
I(L) = \int_0^L \frac{dx}{c} \approx \sum_{e=1}^{NE} \frac{\Delta x^e}{\bar{c}^e}
\]  
(27)

where \(NE\) is the number of elements, \(\bar{c}^e = \sqrt{E^e/\rho}\) is the velocity of propagation of the wave at element \(e\), and

\[
E^e = \begin{cases} 
E^e & \text{if } \sigma < 0 \text{ compression;} \\
E^e_{eff} & \text{if } \sigma \geq 0 \text{ tension.}
\end{cases}
\]  
(28)

ii) Computation of the new time step

\[
\Delta t = \frac{I(L)}{NE}.
\]  
(29)

iii) Find the \(l^{th}\) node coordinate from

\[
\int_0^x \frac{dx'}{c} = l\Delta t
\]

\[
x_l = \frac{l\Delta t - I(j)}{I(x_{j+1}) - I(x_j)} (x_{j+1} - x_j) + x_j.
\]  
(30)

Calculation of displacements and velocities at the new node sites by linear interpolation, the indices \(j\) and \(i\) refer to old and new node positions,

\[
d_i = d_j + (d_{j+1} - d_j) \frac{x_j - x_i}{x_{i+1} - x_i}
\]

\[
v_i = v_j + (v_{j+1} - v_j) \frac{x_j - x_i}{x_{i+1} - x_i}.
\]  
(31)
iv) Determination by equilibrium of the accelerations at the new node sites

\[ a_i = \frac{(f_i^{int} + f_i^{ext})}{m_i}. \]  

(32)

v) Update of the internal variables, \( a \) and \( \varepsilon^T \), for the newly defined elements (see Fig.5), using a length average rule,

\[ a(e^i) = [(x_{j+1} - x_i) \ a(e^j) + \sum_{k=j+1}^{j+n-1} (x_{k+1} - x_k) \ a(e^k) + \\
+ (x_{i+1} - x_{j+n-1}) \ a(e^{j+n})] \frac{1}{(x_{i+1} - x_i)}, \]  

(33)

where \( e^j \) is the \( j \)th element, which lies between the coordinates \( x_{j+1} \) and \( x_j \).
A particular case in this procedure is found when the new element nodes lie fully inside an element of the previous mesh, in this case $a(e^i) = a(e^j)$. Similar expressions apply to $\varepsilon^T(e^i)$.

After remeshing, time integration can restart.

IV. Numerical Integration of the Constitutive Law

A fully implicit scheme for the integration of the constitutive equation is used. It is known that the method is unconditionally stable and consequently no additional time-step constraint is added other than that required by accuracy. Since an explicit scheme is employed in the solution of the wave propagation problem, the time step increment that arises from stability requirements at that level is small enough to guarantee good accuracy in the time integration of the constitutive relations.

The steps in the integration of the constitutive law are:

i) Given $\sigma_n$, $\varepsilon_n$, $a_n$ and $\varepsilon^T_n$, compute
\begin{equation}
    f = c_R \left( 1 - \left( \frac{K_{IC}}{2\sigma_n} \right)^m \left( \frac{\pi}{a_n} \right)^{\frac{m}{2}} \right). \tag{34}
\end{equation}

If \( f \leq 0 \) no update is necessary since the applied stress cannot produce crack propagation.

ii) Solve the non-linear equation in \( a_{n+1} \), viz.

\[
    a_{n+1} - a_n - f(a_{n+1}, \sigma_{n+1}) \Delta t = 0
\]

\[
    \sigma_{n+1} = (\varepsilon_n - \varepsilon_{n+1}^T) E_{eff} (a_{n+1})
\]

\[
    \frac{1}{E_{eff}} = \frac{1}{E_s} - \frac{2\nu^2}{1-\nu} \frac{1}{E}
\]

\[
    \frac{1}{E_s} = \frac{1}{E} + \frac{16}{3} \frac{(1-\nu^2)N a_{n+1}^3}{E}
\]

\[
    \varepsilon_{n+1}^T = \frac{16}{3} \frac{(1-\nu^2)}{E} N (a_{n+1}^3 \sigma_R),
\]

by an iterative procedure, e.g. Newton-Raphson.

iii) Update the main variables \( \sigma_{n+1}, a_{n+1} \) and \( \varepsilon_{n+1}^T \).
CHAPTER III

Applications

I. A Contact Problem

In order to show the general capabilities of the method and to assess the importance of relation (26), a contact problem of two dissimilar bars is considered. The geometry of the problem and the parameters used in the analysis are given in Fig. 6 (see Hughes et al. [13]).

In addition to the steps given in the description of the explicit-implicit algorithm, consideration of contact requires the following:

i) Determination of whether contact is taking place, i.e. if $|y^- - y^+| > \text{tolerance}$,

\[
\begin{align*}
    y^- &= x^- + d^- \\
    y^+ &= x^+ + d^+,
\end{align*}
\]  

(36)

where the plus and minus signs refer to field quantities at the two surfaces that can be in contact.

ii) In case of contact one introduces compatibility of displacements and velocities according to the following mass averaging formulae

\[
\begin{align*}
    d_i &= \frac{m(e^i) \ y^- + m(e^{i+1}) \ y^+}{m(e^i) + m(e^{i+1})} - x_i \\
    d_{i+1} &= \frac{m(e^i) \ y^- + m(e^{i+1}) \ y^+}{m(e^i) + m(e^{i+1})} - x_{i+1},
\end{align*}
\]  

(37.a)
Figure 6. Impact of two dissimilar bars. Velocity-Time profile for Courant number \( r = 1 \).

\[
\begin{align*}
    v_i &= \frac{m(e^i) v_i + m(e^{i+1}) v_{i+1}}{m(e^i) + m(e^{i+1})} \\
    v_{i+1} &= v_i, \\
\end{align*}
\]

(37.b)

where \( e^i \) and \( e^{i+1} \) are the two elements in contact, and the indices \( i \) and \( i+1 \) refer to the nodes located on the contact surface.
Figure 7. Velocity-Time profiles for Courant number $r \neq 1$. 

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iii) Update the nodal accelerations from equilibrium

\[ a_i = \frac{f_i + f_{i+1}}{m(e^i) + m(e^{i+1})} \]

\[ a_{i+1} = a_i \]

\[ f_i = f_i^{int} + f_i^{ext}. \] (38)

For a Courant number \( r \neq 1 \), a comparison between the results for \( \beta \) computed according to relation (26) and for an arbitrary choice is shown in Fig. 7. Although in the former case no spurious oscillations appear, in the latter case, high frequency oscillations are developed.

II. Bar Subjected to Two Symmetric Pulses

In the past, considerable effort has been devoted to the understanding of pathological behavior in computations associated with strain-softening materials. When strain-softening, rate-independent materials are considered in dynamic problems, the governing equations change character from hyperbolic to elliptic, the wave speed becomes imaginary (Hadamard [10]) and the subsequent boundary value problem is ill-posed. Sandler and Wright [22] pointed out the implications of this fact in numerical calculations. They showed that a bar with strain-softening material is very sensitive to the boundary conditions and the size of the elements. They also concluded that rate-dependent models of strain-softening are stable in the sense of Hadamard, but in all cases physical instabilities are observed. Shawki [23] showed that weak strain-softening with strong rate sensitivity leads to stable deformations. Needleman [15] has shown that material rate-dependence eliminates the pathological mesh size effects by implicitly introducing a length scale into the governing
equations, even if the constitutive description does not contain a parameter with the dimension of length.

One encounters the aforementioned scenario in the present work. The constitutive equations introduced in Chapter I are rate dependent; additionally, no imaginary wave speed is possible since the damage model is based on a secant modulus which is always greater than zero. Consequently, one expects a well-posed boundary value problem. In order to show that this is the case, let us consider a bar subjected to two tensile pulses travelling from the ends of the bar to the center, with stress magnitudes that can only exceed the threshold stress $\sigma_c$ when the pulses are superposed. This problem has been considered by Bazant and Belytschko [1]. The description of the geometry and applied loads are given in Figure 8. The parameters used in the analysis are provided in Figure 15.b.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig8.png}
\caption{Bar subjected to two symmetric tensile pulses.}
\end{figure}

Data:
\begin{align*}
\sigma_0 &= 400 \text{ MPa} & t_p &= 30 \text{ nsec} \\
A &= 1 \text{ m} & \text{(area)}
\end{align*}

A tensile force is applied at time $t = 0$. The initial accelerations at the boundaries are computed by equilibrium. It should be mentioned that spurious oscillations are avoided if, in the process of unloading, the corresponding accelerations are also removed.
Figure 9.a. Position-crack radius at different times and number of elements.
Figure 9.b. Position-crack radius at different times and number of elements.
Figure 10.a. Position-element size at different times and number of elements.
Figure 10.b. Position-element size at different times and number of elements.
Figure 11.a. Velocity-Time profiles for different number of elements.
Figure 11.b. Velocity-Time profiles for different number of elements.
Figure 12.a. Stress-Time profiles for different number of elements.
Figure 12.b. Stress-Time profiles for different number of elements.
Figure 13. Remeshing vs. No-remeshing velocity-time profiles.
Figure 14. Remeshing vs. No-remeshing stress-time profiles.
Figures 9.a and 9.b show the crack radius \( a \), a measure of damage in the model, as a function of position in the bar for different times and number of elements. One notes that the magnitude and the size of the damage zone show only slight variations with mesh refinement. Moreover, the energy dissipation associated with the damage process remains almost invariant; lack of invariance is a major drawback in local strain-softening models (Bazant and Belytschko [1]). Figures 10.a and 10.b show the element sizes at different times when the mesh is refined from 151 to 901 elements.

The velocity-time and stress-time profiles are given in Figures 11 and 12. The degree of resolution for the particle velocity increases drastically with the number of elements. It can be said that a good characterization of the solution is obtained for 301 elements. Part of the observed differences are due to the change in the rate of the applied load.

To close this section, the influence of remeshing is emphasized in Figures 13 and 14. High frequency oscillations in stresses are observed when remeshing is not performed. These are mainly due to the fact that the wave does not travel one element per time step after damage takes place. The oscillations are manifested in the central region after the tensile pulses are reflected from the free ends of the bar as compressive pulses. The parameters used in this analysis are: \( K_{IC} = 1 \ MPa\sqrt{m} \), \( N = 0.5\epsilon_{13} \ 1/m^3 \), \( a_0 = 3 \ \mu m \), \( \sigma_R = 0 \), and \( m = 1 \).

### III. Results for the Experimental Setup

Three shots have been analyzed, corresponding to the successful experimental results. The material properties, shot characteristics, and experimentally measured
velocity-time profiles are reproduced after G. Raiser et al. [20] in Figures 15.a and 16, respectively. In Figure 15.b the values of the parameters used in the numerical simulation are given.

Damage initiates at the section where tensile stresses are first developed, point D in Figure 1. It is followed by an attenuation of stresses. The attenuation is characterized by a decay at the wave front of the applied stress \(\sigma_0\) to a diminished stress level \(\sigma_c\). The fully decayed stress \(\sigma_c\) is the stress necessary to produce crack growth according to equation (14). The time that it takes the stress to decay from \(\sigma_0\) to \(\sigma_c\) can be defined as the characteristic time \(t_c\). According to the precursor analysis given in [20], the threshold stress \(\sigma_c\) and the characteristic time \(t_c\) are given by

\[
\sigma_c = K_{IC}/2\sqrt{a_0/\pi}
\]

\[
t_c = \frac{4}{3} \frac{1 + g(\nu)Na_0^3}{g(\nu)Na_0^2c_R}
\]

\[
g(\nu) = \frac{16}{3} \frac{(1 - \nu^2)(1 - \nu)}{1 - \nu - 2\nu^2}.
\]

Figure 17 shows the computed velocity-time profiles for the three shots. Good agreement with the main features of the experiment is seen. These main features are:

- the characteristic time \(t_c\), measured indirectly by the duration of the tail at the end of the first pulse,

- the level of attenuation and the spreading of the second pulse,

- the slope of the stress-time profile behind the wave front in the second pulse.
<table>
<thead>
<tr>
<th>Material</th>
<th>Longitudinal Wave Speed $c_1$, mm/μs</th>
<th>Transverse Wave Speed $c_2$, mm/μs</th>
<th>Acoustic Impedance, GPa/mm/μs</th>
<th>Shear Impedance, GPa/mm/μs</th>
<th>Mass Density $\rho$, Kg/m³</th>
</tr>
</thead>
<tbody>
<tr>
<td>1018 CR Steel</td>
<td>5.9</td>
<td>3.2</td>
<td>45.5</td>
<td>24.7</td>
<td>7700</td>
</tr>
<tr>
<td>6061-T6 Aluminum</td>
<td>6.4</td>
<td>3.05</td>
<td>17.34</td>
<td>8.66</td>
<td>2700</td>
</tr>
<tr>
<td>Alumina Ceramic</td>
<td>10.8</td>
<td>6.4</td>
<td>43.09</td>
<td>25.54</td>
<td>3990</td>
</tr>
</tbody>
</table>

**Figure 15.a.** Material Properties.

<table>
<thead>
<tr>
<th>Shot No.</th>
<th>Projectile Velocity $V_0$, mm/μs</th>
<th>Interface Stress $\sigma$, MPa</th>
<th>Tilt $\theta$, mrad</th>
<th>Flyer Thickness, mm</th>
<th>Specimen Thickness, mm</th>
<th>Momentum Trap Thickness, mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>87-02</td>
<td>0.0521</td>
<td>661</td>
<td>0.96</td>
<td>0.90</td>
<td>4.00</td>
<td>3.10</td>
</tr>
<tr>
<td>87-04</td>
<td>0.0710</td>
<td>902</td>
<td>1.00</td>
<td>0.86</td>
<td>4.00</td>
<td>3.07</td>
</tr>
<tr>
<td>88-04</td>
<td>0.0484</td>
<td>614</td>
<td>0.46</td>
<td>0.88</td>
<td>4.00</td>
<td>2.52</td>
</tr>
</tbody>
</table>

**Figure 15.b.** Summary of Experiments. Parameters used in the numerical simulation.

In Figure 18, stress-strain plots at the fixed position $x = 2.1$ mm are provided. The peak stress, permanent strain, and some indication of the damage
evolution characterized by the secant modulus are shown. Small discrepancies can be observed when reloading in compression occurs; these are mainly due to the remeshing procedure.

Figures 19-21 provide some details of the nature of the propagating waves in the specimen (as reference see Fig. 1 corresponding to the elastic solution). Stress levels are shaded in a position-time diagram. One can appreciate the damage initiation zone, followed by an attenuation of stress, together with the spread of the pulse. The last effect is due to the delay of the wave front generated by the reduction of the wave speed in the damage zone. In addition one notes the reflection produced in the damage zone upon the arrival of the second compressive pulse. This effect is observed only if permanent strain in the material is allowed to take place. Another feature that can be extracted from these plots is the compressive waves caused by the inelastic strain rate developing in the damage zone. This feature is observed as a tail of the first compressive pulse according to the above description of the velocity-time profiles. Figure 22 is a magnification of the damage zone in the (t-x) diagram that shows the crack radius evolution in time for the three shots under study. The boundary of these zones, shaded blue, correspond to the initial value of the microcrack radius. Increasing values of \(a\) are represented by different colors of the spectrum. It can be mentioned that while in shots 87-04 and 87-02 the plots are non symmetric, in shot 88-04 a quasi-symmetry is observed. This is easily explained from the fact that the width of the tensile pulse is bigger for this shot than for the other two. As a result, in shot 88-04 the damage zone grows forward and backward at a similar rate, with respect to the section where tensile stresses are first achieved (point D in Fig. 1).

In order to have a good understanding of the capabilities and limitations of the model, a parametric study is carried for shot 87-04. A standard set of parameters,
Figure 16. Velocity-Time Profiles at Rear Surface of Momentum Trap. Experimental results [20].
Figure 17. Velocity-Time Profiles at Rear Surface of Momentum Trap. Numerical Solution.
Figure 18. Stress-Strain Curves at $z = 2.1 \text{ mm}$. 

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FIGURE 20. STRESS AS A FUNCTION OF POSITION AND TIME.
FIGURE 21. STRESS AS A FUNCTION OF POSITION AND TIME.
Figure 22. Damage evolution in time for Shots 87-04, 87-02 and 88-04 respectively.
given in Fig. 15.b, is used as reference. The extremum values of these parameters are considered. Figure 23 shows the velocity-time profiles for different values of the microcrack density. It can be inferred that $N$ has an strong influence in the shape and dimension of the tail at the end of the first pulse, as well as in the spreading of the second pulse. By contrast, no appreciable influence is seen in the slope behind the wave front in the second pulse. The same can be said of all the other parameters with the exception of $m$. In fact, this provides the motivation for the introduction of this additional parameter to allow some degree of control on the rate at which damage occurs, see Fig. 24. The value $m = 1$ closely corresponds to the universal function given by Freund [8] for a semi-infinite crack. As expected, the tail of the pulse and the threshold stress $\sigma_c$ are strongly dependent on the value of $m$. In Fig. 25 the main role of $K_{IC}$ in governing the threshold stress $\sigma_c$ is shown. Moreover, its influence on the characteristic time $t_c$ can be seen. Figure 26 exhibits the effect of the residual stress $\sigma_R$. While the $t_c$ and $\sigma_c$ values remain almost unchanged, a strong effect in the spread of the tensile pulse is observed. It should be also noted that the introduction of residual stresses almost eliminates unrealistic negative velocities. Finally, the influence of the initial microcrack radius $a_0$ is plotted in Fig. 27. Its main effect seems to be in the threshold stress $\sigma_c$.

Equation (39.a) matches consistently the numerical results. Difficulties are encountered if one attempts to correlate $t_c$ from (39.b) with the duration of the tail of the compressive pulse. In fact for the case under analysis, $K_{IC} = 1$ $MPa\sqrt{m}$, $\nu = 0.22$, $c_R = 5838$ $m/sec$ and $N = 0.5e14$ $1/m^3$ one obtains:

$$a_0 = 1.5 \mu m \quad t_c = 350.7 \ nsec \quad \sigma_c = 723 \ MPa$$

$$a_0 = 4.0 \mu m \quad t_c = 50.2 \ nsec \quad \sigma_c = 443 \ MPa$$

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Note that according to (39.b) there is a sharp decrease in $t_c$ with increasing $a_0$. By contrast, the duration of the tail of the compressive pulse in Figure 27 remains ostensibly unchanged. A possible explanation may be obtained from the fact that the inelastic deformation rate increases strongly behind the wave front, a fact not brought to light by the precursor analysis which is carried on at the wave front.
Figure 23. Velocity-Time profiles for different values of the crack density.
Figure 24. Velocity-Time profiles for different values of the exponent m.
Figure 25. Velocity-Time profiles for different values of the fracture toughness.
Figure 26. Velocity-Time profiles for different values of the residual stress.
Figure 27. Velocity-Time profiles for different values of the initial crack radius.
CONCLUSIONS

The proposed model ensures well-posed boundary value problems in contrast to the difficulties that arise when strain-softening rate independent materials are considered in dynamic problems. The constitutive equation, developed in this thesis, is rate dependent and therefore no pathological mesh size dependencies are observed, since the rate dependence implicitly introduces a length scale into the governing equations.

The model captures many of the relevant features observed in the experiments. Nonetheless, several limitations apply. First, no damage in compression is considered. Even while the tensile damage far exceeds that which develops under compression, the attenuation of the normal velocity at the rear surface of the specimen in the first pulse, is a clear indication of compressive damage, see Figure 16. Secondly, a dilute set of microcracks is envisioned; other possibilities can be explored within the framework of self-consistent models. Moreover, the microcracks are assumed to be oriented perpendicular to the direction of the applied load. Transmission electron micrographs of recovered specimens indicate that the microcracks occur along grain boundaries and are not oriented, exclusively, to be perpendicular to the applied tensile stress (see [20]). If damage in compression were to be considered, additional orientations would need to be taken into consideration to account for mechanisms such sliding of crack surfaces. Finally, no elastic dispersion, due to scattering of the waves at the cracks has been considered in the analysis. These refinements are to be pursued in future work.
References


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